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Interpolation to Continuous Data on Curves with Corners*

MAYNARD THOMPSON

Department of Mathematics, Indiana University, Bloomington, Indiana 47401
and
University of Maryland, College Park, Maryland 20742

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1. INTRODUCTION

Let Γ be a Jordan curve in the finite z-plane whose interior and exterior are denoted by $\operatorname{Int}\Gamma$ and $\operatorname{Ext}\Gamma$, respectively, and let $\mathscr{F}$ be a family of complex-valued functions defined on Γ. The following question has a long history in approximation theory: Find a sequence of point sets $\{S_n\}_{n=1}^{\infty}$, where $S_n = \{z_{nk}\}_{k=1}^{n}$ consists of n (distinct) points on Γ, such that the polynomials $L_n[f; \cdot]$ of degree at most $n-1$ which interpolate to a fixed but arbitrary function $f \in \mathscr{F}$ on S_n converge on $\operatorname{Int}\Gamma$. Implicit in this is the assumption that the limit function $F = \lim_{n\to\infty} L_n[f; \cdot]$ has an appropriate relation to f.

Present knowledge is satisfactory only in the case where $\mathscr{F}$ is the family of functions analytic on $\Gamma \cup \operatorname{Int}\Gamma$. For this case, Fejér [8, Section 7.6] proved that $\lim_{n\to\infty} L_n[f; z] = f(z)$, uniformly on $\Gamma \cup \operatorname{Int}\Gamma$, if and only if the sets S_n are uniformly distributed on Γ. We refer to [8] for the definition of uniform distribution. Curtiss [1–3] has studied this question in the case where $\mathscr{F} = \mathscr{C}(\Gamma)$, the family of functions continuous on Γ. He proved that if Γ is sufficiently smooth and if the sets S_n consist of the images of the roots of unity under a suitable mapping function, then for $z \in \operatorname{Int}\Gamma$ one has

$$\lim_{n\to\infty} L_n[f; z] = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{1}$$

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In [6] the author indicated permissible perturbations of the sets S_n which maintain the conclusion of Curtiss's theorem. In this paper we consider the question for $\mathscr{F} = \mathscr{C}(\Gamma)$, Γ being a piecewise smooth curve with corners of certain types. The only previous results for nonsmooth Γ and $\mathscr{C}(\Gamma)$ are in [5] where only certain cusps were allowed. We refer the reader to [2] and [8] for surveys of the problem and its history.

2. Definitions, Notation and Preliminaries

We suppose that the origin of the z-plane is contained in Int Γ and that the logarithmic capacity of Γ is 1. Let $z = \Phi(w)$ be the one-to-one conformal mapping of the exterior of the unit disc in the w-plane onto Ext Γ, normalized by the conditions $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. There is a natural parametrization ϕ of Γ obtained by extending Φ to a homeomorphism of $|w| \geqslant 1$ onto $\Gamma \cup \operatorname{Ext} \Gamma$ and setting $\phi(t) = \Phi(e^{2\pi it})$, $0 \leqslant t \leqslant 1$. Extend the domain of definition of ϕ to all real values of t by periodicity.

In this paper we consider a relaxation of the smoothness conditions imposed by Curtiss. In particular, suppose that Γ possesses a continuously turning tangent except at a finite number of points $p_1, \ldots, p_K$, at which we suppose Γ to have half tangents with interior angle $\pi\beta_j$, $0 < \beta_j < 2$, $j = 1, 2, \ldots, K$. For each j, the region Ω_j in the ζ-plane defined by

$$\Omega_j = \left\{ \zeta : \zeta = \frac{1}{z} - \frac{1}{p_j}, \, z \in \operatorname{Ext} \Gamma \right\}$$

is a bounded simply connected region not containing the origin. Thus there is a single valued branch of $\zeta^{1/(2-\beta_j)}$ defined on Ω_j. Introduce

$$\Theta(w) = \left\{ \frac{1}{\Phi(w)} - \frac{1}{p_j} \right\}^{1/(2-\beta_j)} \tag{2}$$

and note that $\mathfrak{z} = \Theta(w)$ maps $|w| > 1$ one-to-one and conformally onto a Jordan region Ω^* in the $\mathfrak{z}$-plane. Moreover, the origin of the $\mathfrak{z}$-plane is a boundary point of Ω^* and $\partial\Omega^*$ has a tangent there. As before, Θ can be extended to $|w| \geqslant 1$ and the function

$$\theta(t) = \Theta(e^{2\pi it}), \qquad 0 \leqslant t \leqslant 1 \tag{3}$$

provides a parametrization of $\partial\Omega^*$. We suppose θ to be defined for all values of t by periodicity.

DEFINITION 1. Γ is said to possess an *admissible corner* at p_j if

(i) $0 < \beta_j < 1$,

(ii) there exists a constant $\eta > 0$ such that

$$\theta(t) = c(t + th(t)), \qquad |t| < \eta,$$

where h is continuously differentiable (as a function of t) in $|t| < \eta$, $h(0) = 0$, and h' has a modulus of continuity ω which satisfies

$$\int_0^\eta \frac{\omega(u)}{u}\, du < \infty.$$

DEFINITION 2. A Jordan curve is said to be *admissible* if

(i) it has a continuously turning tangent except at a finite number of admissible corners $\{p_j\}$,

(ii) there is an integer N and a real number τ such that

$$\{p_j\} \subset \{z : z = \phi[(k/N) + \tau], k = 1, 2,..., N\}.$$

Clearly these conditions are of quite different types. The first specifies the local behavior of the curve in a neighborhood of a corner and the second stipulates how the corners are distributed over the curve. Set inclusion in (ii) may be proper, that is, the number of corners may be much smaller than N. A number of geometric conditions on Γ which guarantee its admissibility can be deduced from theorems of Lindelöf, Ostrowski, and Warschawski [4, 7]. As examples of admissible curves Γ we cite any regular polygon, and the curve Γ specified by

$$\Gamma = \{z : |z(z-2)| = 1, \operatorname{Re} z \leqslant 1\}.$$

Let $\mathscr{S}$ be the set of preimages on the unit interval of the corners $\{p_j\}$, i.e., $\mathscr{S} = \{s : 0 \leqslant s < 1, \phi(s) \in \{p_j\}\}$. Then for $s \in [0, 1) \setminus \mathscr{S}$ define

$$\Psi(s, w) = \begin{cases} [\Phi(w) - \phi(s)]/(w - e^{2\pi i s}), & |w| \geqslant 1, \quad w \neq e^{2\pi i s}, \\ \Phi'(e^{2\pi i s}), & w = e^{2\pi i s}, \end{cases} \tag{4}$$

and

$$\psi(s, t) = |\Psi(e^{2\pi i t})|, \qquad 0 \leqslant t \leqslant 1. \tag{5}$$

Extend ψ by periodicity to all real t. The following lemma can be proved by an application of the Cauchy integral theorem to the function $w^{-1} \log \Psi(s, w)$ which is holomorphic in $|w| > 1$ and continuous on $|w| \geqslant 1$ (cf. [1]). Here that branch of the logarithm is selected for which $\lim_{|w| \to \infty} \log \Psi(s, w) = 0$.

LEMMA 1. *If* $s \in [0, 1) \setminus \mathscr{S}$ *then* $\int_0^1 \log \psi(s, t)\, dt = 0$.

Let $S_n = \{z_{nk}\}_{k=1}^n$ and form the fundamental polynomials $\omega_n(z) = \prod_{k=1}^n (z - z_{nk})$. The Lagrange polynomial $L_n[f; \cdot]$ which interpolates on the set S_n to a function f defined on Γ is given by

$$L_n[f; z] = \sum_{k=1}^n \frac{f(z_{nk})\, \omega_n(z)}{\omega_n'(z_{nk})(z - z_{nk})}.$$

Following Curtiss [2] we are led to consider, for $z \in \operatorname{Int} \Gamma$,

$$\lim_{n\to\infty} \omega_n(z), \tag{6}$$

and

$$\| L_n[\cdot; z] \| = \sum_{k=1}^n | \omega_n(z)| \, | \omega_n'(z_{nk})(z - z_{nk})|^{-1}. \tag{7}$$

Known techniques [1, 6] give

LEMMA 2. *If* Γ *is admissible and*

$$S_n = \left\{\phi \left(\frac{k}{n} + \tau_n\right)\right\}_{k=1}^n,$$

where τ_n *are real numbers depending on n but not on k, then* $\lim_{n\to\infty} \omega_n(z) = 1$ *uniformly on closed subsets of* $\operatorname{Int} \Gamma$.

It follows from [2] that for S_n as in Lemma 2 a sufficient condition for (1) to hold uniformly on closed subsets of $\operatorname{Int} \Gamma$ for any $f \in \mathscr{C}(\Gamma)$ is that the sequence of norms $\{\| L_n[\cdot; z]\|\}_{n=1}^\infty$ be uniformly bounded for z in any closed subset of $\operatorname{Int} \Gamma$. The major portion of this paper is devoted to establishing this boundedness.

3. APPROXIMATE QUADRATURE AND OTHER LEMMAS

In this section we collect some technical lemmas which will be used later. The first two can be viewed as error estimates for particular numerical integration schemes. Lemma 3 is well known [1] and is stated here only for completeness. The second is more interesting. It provides a uniform estimate for the error in approximate quadrature of a family of functions each of which has no more than a logarithmic singularity.

LEMMA 3. *Let f be absolutely continuous on $[a, b]$ and let n be a positive integer. Then*

$$\left| \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) - n \int_0^1 f(t)\, dt \right| \leqslant \int_a^b |f'(t)|\, dt.$$

LEMMA 4. *Let f be defined on the rectangle $0 \leqslant s \leqslant 1$, $a \leqslant t \leqslant b$ with the point $(s, t) = (0, a)$ deleted by*

$$f(s, t) = \log(s + t - a).$$

For each positive integer n set $t_{nk} = a + (k - \frac{1}{2})(b - a)/n$, $k = 1, 2,..., n$. Then for all s, $0 \leqslant s \leqslant 1$, and for $n = 1, 2,...,$

$$\left| \sum_{k=1}^{n} f(s, t_{nk}) - n \int_a^b f(s, t)\, dt \right| \leqslant 1. \tag{8}$$

Proof. Making a change of scale on the t-axis if necessary, we may assume $a = 0$ and $b = 1$. Straightforward integration gives

$$\int_0^1 \log(s + t)\, dt = \begin{cases} (1 + s)\log(1 + s) - s \log s - 1, & 0 < s \leqslant 1, \\ -1, & s = 0. \end{cases} \tag{9}$$

We prove (8) for $s > 0$, and since the left-hand side is a continuous function of s at $s = 0$, the desired inequality holds for $s = 0$ as well. We begin by observing that

$$n[(1 + s)\log(1 + s) - s\log s - 1 + \log n]$$
$$= \sum_{k=1}^{n} [(k + ns)\log(k + ns) - (k - 1 + ns)\log(k - 1 + ns) - 1],$$

and using this together with the explicit form of t_{nk} in the left-hand side of (8), which we denote by $R(s)$, we obtain

$$\begin{aligned} R(s) &= \left| \sum_{k=1}^{n} \log\left(s + \frac{k - \frac{1}{2}}{n}\right) - n[(1 + s)\log(1 + s) - s\log s - 1] \right| \\ &= \left| \sum_{k=1}^{n} \left\{ \log\left(\frac{k - \frac{1}{2} + ns}{k + ns}\right) - (k - 1 + ns)\log\left(\frac{k + ns}{k - 1 + ns}\right) + 1 \right\} \right| \\ &\leqslant \left| \sum_{k=2}^{n} \left\{ \log\left(1 + \frac{1}{2k - 1 + 2ns}\right) \right. \right. \\ &\qquad \left. \left. + (k - 1 + ns)\log\left(1 + \frac{1}{k - 1 + ns}\right) - 1 \right\} \right| + \epsilon(n, s). \end{aligned}$$

Here $\epsilon(n, s)$ is the term corresponding to $k = 1$ in the second sum; it can be bounded by routine methods. Each of the logarithms occurring in the sum in the last expression can be expanded in a Taylor series and the resulting double sum will have the form

$$\left| \sum_{k=2}^{n} \left\{ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(2k-1+2ns)^j} + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(k-1+ns)^{j-1}} - 1 \right\} \right|. \tag{10}$$

Writing out the series in the curly brackets in (10) and rearranging terms, we have

$$\begin{aligned} &\left(\frac{1}{2k-1+2ns} - \frac{1}{2k-2+2ns}\right) \\ &\qquad + \left(\frac{1}{3(k-1+ns)^2} - \frac{1}{2(2k-1+2ns)^2}\right) - \cdots \\ &\qquad = \frac{-1}{(2k-1+2ns)(2k-2+2ns)} + a_2 - a_3 + a_4 - \cdots. \end{aligned}$$

An easy calculation shows that $a_j \geqslant a_{j+1} > 0$ for all $j \geqslant 2$, and therefore the series $\sum_{j=2}^{\infty} (-1)^j a_j$ converges to a sum A, $0 < A < a_2$. Also $a_2 < [(2k-1+2ns)(k-1+ns)]^{-1}$ so the entire series in curly brackets converges to a sum $A_k(s)$ with

$$| A_k(s) | < \frac{1}{4(k-1+ns)^2} < \frac{1}{4(k-1)^2}, \qquad k \geqslant 2.$$

Using this and easily computed bounds on $\epsilon(n, s)$, we finally have

$$R(s) < \frac{1}{4} \sum_{k=1}^{\infty} k^{-2} + \epsilon(n, s) < 1.$$

The purpose of the next lemma is to use the conditions associated with admissibility to obtain a representation of ψ which is valid for s and t near certain critical values.

LEMMA 5. *Let Γ be an admissible curve with interior angle $\beta_j \pi$ at the corner $p_j = \phi(s_j)$, $j = 1, 2, \ldots, K$. Then for each j, $j = 1, 2, \ldots, K$, there is a positive number $\delta = \delta_j$ and a function $\chi = \chi_j$ defined on*

$$D = \{(s, t): 0 \leqslant | t - s_j | \leqslant \delta, 0 < | s - s_j | \leqslant \delta\}$$

and satisfying

(i) $\psi(s, t) = \chi(s, t)(| t - s_j | + | s - s_j |)^{1-\beta_j}, \qquad (s, t) \in D,$

(ii) *for each s, $0 < |s - s_j| \leqslant \delta$, the function $\chi(s, \cdot)$ is absolutely continuous on* $0 \leqslant |t - s_j| \leqslant \delta$,

(iii) *there are constants $m > 0$, M_1 and M_2 such that*

$$m \leqslant \chi(s, t) \leqslant M_1, \qquad (s, t) \in D,$$

$$\int_{s_j-\delta}^{s_j+\delta} \left| \frac{d\chi}{dt}(s, t) \right| dt \leqslant M_2, \qquad 0 < |s - s_j| \leqslant \delta.$$

Proof. There is no loss of generality, and the computations are easier, if we assume $s_j = 0$. We will show that suitable δ and χ exist for $s > 0$, $t \geqslant 0$. The details for the remaining portions of D are similar. We set $\beta = \beta_j$, $\alpha = 2 - \beta_j$. Then $\alpha\pi$ is the exterior angle at $p_j = p$ and $1 < \alpha < 2$. Since Φ maps $|w| > 1$ onto Ext Γ it is the exterior angle which naturally occurs in our approach to the problem. By (4) and (5) we have for s and t sufficiently small, $s \neq 0$,

$$\psi(s, t) = \begin{cases} \left| \dfrac{\phi(t) - \phi(s)}{e^{2\pi i t} - e^{2\pi i s}} \right|, & t \neq s, \\ |\phi'(s)|/2\pi, & t = s, \end{cases}$$

where the prime denotes differentiation. Henceforth we will consider only the first of these expressions, and assume that the obvious extension is made for $t = s$. Also, by (2) and (3), recalling that $\beta = 2 - \alpha$,

$$\phi(t) = \left\{ [\theta(t)]^\alpha + \frac{1}{p} \right\}^{-1}$$

and consequently

$$\begin{aligned} \psi(s, t) &= \left| \frac{[\theta(t)]^\alpha - [\theta(s)]^\alpha}{\{[\theta(t)]^\alpha + (1/p)\}\{[\theta(s)]^\alpha + (1/p)\}(e^{2\pi i t} - e^{2\pi i s})} \right| \\ &= \left| \frac{[\theta(t)]^\alpha - [\theta(s)]^\alpha}{t - s} \right| g(s, t). \end{aligned}$$

It follows from the properties of θ and the exponential function that for sufficiently small $\delta > 0$ the function $g(s, \cdot)$ is continuously differentiable on $0 \leqslant t \leqslant \delta$ and there are constants m, M_1, M_2 such that $0 < m \leqslant g(s, t) \leqslant M_1$, $0 \leqslant t \leqslant \delta$, $0 < s \leqslant \delta$, and $|(dg/dt)(s, t)| \leqslant M_2$, $0 \leqslant t \leqslant \delta$, $0 < s \leqslant \delta$. The proof of the lemma will be completed by showing that the function

$$G(s, t) = \left| \frac{[\theta(t)]^\alpha - [\theta(s)]^\alpha}{t - s} \right| (s + t)^{1-\alpha}$$

satisfies the conditions required of χ in (ii) and (iii) above. By the admissibility of Γ we have $\theta(t) = c[t + th(t)]$, and hence

$$G(s, t) = |\, c\,|^{\alpha} \left| \frac{t^{\alpha}[1 + h(t)]^{\alpha} - s^{\alpha}[1 + h(s)]^{\alpha}}{t - s} (s + t)^{1-\alpha} \right|$$

$$= |\, c\,|^{\alpha} \left| \left[\frac{t^{\alpha} - s^{\alpha}}{t - s}\right] (t + s)^{1-\alpha}[1 + h(t)]^{\alpha} + \frac{s^{\alpha}}{(s + t)^{\alpha - 1}} \left\{\frac{[1 + h(t)]^{\alpha} - [1 + h(s)]^{\alpha}}{t - s}\right\} \right|. \tag{11}$$

Consider first the function

$$r(s, t) = \left(\frac{t^{\alpha} - s^{\alpha}}{t - s}\right) (t + s)^{1-\alpha}.$$

For each fixed s this function is positive, absolutely continuous, monotonically increasing from 1 to $\alpha 2^{1-\alpha}$ as t increases from 0 to s, and decreasing as t increases from s to δ. The total variation of $r(s, \cdot)$ in $[0, \delta]$ is less than $2[\alpha 2^{1-\alpha} - 1]$ and therefore

$$\int_0^{\delta} \left| \frac{dr}{dt}(s, t) \right| dt < 2(\alpha 2^{1-\alpha} - 1)$$

for all s, $0 < s \leqslant \delta$. Also, h is continuously differentiable on $[0, \delta]$ and $\lim_{t \to 0} h(t) = 0$ which implies that the function $[1 + h(t)]^{\alpha}$ is absolutely continuous, and bounded above and away from zero for t sufficiently small. It follows that the product $r(s, t)[1 + h(t)]^{\alpha}$ has all of the desired properties.

We continue the proof by showing that the remaining term in (11) has the desired properties and, moreover, can be made as small as we choose by selecting δ small enough. Then the sum of both terms in (11) satisfies (ii) and (iii). It is clear that the term $s^{\alpha}(s + t)^{1-\alpha}$ is absolutely continuous, has a uniformly bounded total variation for $0 < s \leqslant \delta$ and can be made small by taking δ small. We turn to the term

$$H(s, t) = \{[1 + h(t)]^{\alpha} - [1 + h(s)]^{\alpha}\}/(t - s).$$

It follows from the differentiability assumption on h that for each fixed s the function $H(s, \cdot)$ is bounded and absolutely continuous on $[0, \delta]$. We show now the existence of a constant M satisfying

$$\int_0^{\delta} \left| \frac{dH}{dt}(s, t) \right| dt < M \tag{12}$$

for all $s \in (0, \delta]$. To this end, set $m(t) = [1 + h(t)]^\alpha$. Since $\lim_{t\to 0} h(t) = 0$ we see that m is differentiable on $[0, \delta]$ for δ sufficiently small. Letting prime denote differentiation, we have

$$H'(s, t) = \left\{\int_s^t [m'(t) - m'(u)]\, du\right\}\Big/(t - s)^2. \tag{13}$$

If ω denotes the modulus of continuity of h' then the modulus of continuity ω_1 of m' satisfies $\omega_1(x) \leqslant K \max[\omega(x), x]$ for some constant K and sufficiently small x. Thus, if $\int_0^\delta \omega(u)/u\, du$ is finite, then so is $\int_0^\delta \omega_1(u)/u\, du$. From (13) we obtain

$$| H'(s, t)| \leqslant \left| \left[\int_0^{|t-s|} \omega_1(u)\, du\right]\Big/(t - s)^2 \right|$$

and consequently

$$\begin{aligned}\int_0^\delta | H'(s, t)|\, dt &\leqslant 2 \int_0^\delta \left[\int_0^x \omega_1(u)\, du\right]\Big/x^2\, dx \\ &\leqslant M_1 + 2 \int_0^\delta \frac{\omega_1(u)}{u}\, du.\end{aligned}$$

Here M_1 is a constant which depends only on ω_1 and δ so that (12) is established and the proof of the lemma is complete.

4. A Theorem for Curves with One Corner

In order to indicate the flavor of the proof and at the same time keep the technical difficulties to a minimum, we consider first a special case. A more general theorem is given in Section 5.

Theorem 1. *Let Γ be an admissible curve with one corner p, and let*

$$S_n = \left\{\phi\left(t_0 + \frac{k - (1/2)}{n}\right)\right\}_{k=1}^n,$$

where $\phi(t_0) = p$. Then for each closed set $E \subset \operatorname{Int} \Gamma$ there is a constant M depending only on E such that $\| L_n[\cdot; z]\| \leqslant M$, $z \in E$.

Proof. Making a shift of coordinates along the t-axis, if necessary, we may take $t_0 = 0$. Introduce $t_{nk} = (k - \frac{1}{2})/n$, $k = 1, 2,..., n$, and adopt the convention that the dependence of t_{nk} on n is understood so that we write simply t_k for t_{nk}. The conclusion of the theorem follows from (7) and Lemma 2 together with the estimate

$$\sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} | \phi(t_j) - \phi(t_k)|^{-1} \leqslant M, \qquad n = 1, 2,..., \tag{14}$$

M a constant. We actually prove something stronger than (14). Indeed, since

$$\prod_{\substack{j=1 \\ j \neq k}}^{n} | e^{2\pi i t_j} - e^{2\pi i t_k} | = n, \qquad k = 1, 2,..., n$$

for all n, the estimate (14) follows if we can show that there is a constant M_1 satisfying

$$\prod_{\substack{j=1 \\ j \neq k}}^{n} \left| \frac{e^{2\pi i t_j} - e^{2\pi i t_k}}{\phi(t_j) - \phi(t_k)} \right| \leqslant M_1 \tag{15}$$

for $k = 1, 2,..., n$ and all n. The remainder of this section is devoted to proving (15).

Taking logarithms and using the notation of (5) we seek to prove that

$$\sum_{\substack{i=1 \\ j \neq k}}^{n} \log \psi(t_k, t_j) \tag{16}$$

is uniformly bounded from below. The technique used to establish this is a standard one of obtaining bounds on

$$I_n(t_k) = \sum_{\substack{j=1 \\ j \neq k}}^{n} \log \psi(t_k, t_j) - n \int_0^1 \log \psi(t_k, t)\, dt.$$

We estimate $I_n(t_k)$ differently depending upon the location of t_k. Take δ, $\delta < 1/2$, such that the conclusions of Lemma 5 hold. If $t_k \in [\delta, 1 - \delta]$ then $\psi(t_k, \cdot) \in C^1$ and is bounded away from zero. For such t_k's the uniform boundedness of $I_n(t_k)$ follows from Lemma 3. The remaining cases, $t_k \in (0, \delta)$, $t_k \in (1 - \delta, 1)$, recall $t_k \neq 0$, require additional effort. The details are given only for $t_k \in (0, \delta)$, the other situation can be handled similarly.

Suppose then $t_k \in (0, \delta)$. Write $I_n(t_k)$ as the sum of three terms: The first consisting of the sum over those j, $j \neq k$, for which $0 < t_j < \delta$ and the integral from 0 to δ; the second consisting of the sum over those j for which $\delta \leqslant t_j \leqslant 1 - \delta$ and the integral from δ to $1 - \delta$; and the third consisting of the remaining terms. Denote these three parts $I_n'(t_k)$, $I_n''(t_k)$ and $I_n'''(t_k)$, respectively. We consider the second one, $I_n''(t_k)$. Suppose there are $m = m(n)$ nodes t_j in the interval $[\delta, 1 - \delta]$; denote them $\tau_1, \ldots, \tau_m$. The expression

$$\left| I_n''(t_k) - \left[\sum_{p=1}^{m} \log \psi(t_k, \tau_p) - \frac{m}{\tau - \tau_1 + (1/n)} \int_{\tau_1-(1/2n)}^{\tau_m+(1/2n)} \log \psi(t_k, t)\, dt \right] \right|$$

is bounded, and an application of Lemma 3 shows that the quantity in square brackets is also uniformly bounded. Thus $I_n''(t_k)$ is uniformly bounded for $t_k \in (0, \delta)$ and all n.

Next we consider $I_n'(t_k)$; the argument for $I_n'''(t_k)$ requires no essentially different techniques. If $\{t_1, \ldots, t_m\} = \{t_j : 0 < t_j < \delta\}$ then $I_n''(t_k) + \log \psi(t_k, t_k)$ differs from

$$A_n(t_k) = \sum_{j=1}^{m} \log \psi(t_k, t_j) - \frac{m}{(m/n)} \int_0^{m/n} \log \psi(t_k, t)\, dt$$

by a bounded quantity. Thus we turn our attention to $A_n(t_k)$, and make use of Lemma 5. If $\alpha\pi$ is the exterior angle at p, we have

$$A_n(t_k) = \left\{ \sum_{j=1}^{m} \log \chi(t_k, t_j) - \frac{m}{(m/n)} \int_0^{m/n} \log \chi(t_k, t)\, dt \right\}$$
$$+ (\alpha - 1) \left\{ \sum_{j=1}^{m} \log(t_k + t_j) - \frac{m}{(m/n)} \int_0^{m/n} \log(t_k + t)\, dt \right\},$$

and the expression in the first curly brackets is bounded (uniformly in t_k and n) by Lemma 3 and the properties of χ, while the expression in the second curly brackets is bounded (uniformly in t_k and n) by Lemma 4.

Finally, by Lemma 1, $\int_0^1 \log \psi(t_k, t)\, dt = 0$ and therefore we have proved that there is a constant M, independent of $t_k \in (0, \delta)$ and n, such that

$$\left| \sum_{\substack{j=1 \\ j \neq k}}^{n} \log \psi(t_k, t_j) + (\alpha - 1) \log 2t_k \right| \leqslant M.$$

Here we have also used the uniform boundedness (in t_k) of $\log \chi(t_k, t_k)$. The analogous estimates for other t_k's are

$$\left| \sum_{\substack{j=1 \\ j \neq k}}^{n} \log \psi(t_k, t_j) \right| \leqslant M, \qquad t_k \in [\delta, 1 - \delta],$$

$$\left| \sum_{\substack{j=1 \\ j \neq k}}^{n} \log \psi(t_k, t_j) + (\alpha - 1) \log 2(1 - t_k) \right| \leqslant M, \qquad t_k \in (1 - \delta, 1).$$

Noting that $1 < \alpha < 2$ and $\delta < 1/2$, we have $(\alpha - 1)\log 2t_k$ and $(\alpha - 1)\log 2(1 - t_k)$ negative, and consequently these three inequalities prove that (16) is bounded below, uniformly in t_k and n. Our proof is complete.

5. Curves with Several Corners

Let Γ be a general admissible curve. One proves the uniform boundedness, on closed subsets of Int Γ, of the Lagrange interpolation operators $L_n[\cdot; z]$ in a manner similar to that used in the proof of Theorem 1. It is necessary to subdivide the interval [0, 1] into several subintervals, each containing only one preimage of a corner, and proceed as above. If the set of corners $\{p_j\}$ is contained in $\{z: z = \phi[(k/N) + \tau], k = 1, 2, 3,..., N\}$, as it must be for some τ and N by admissibility, then we take

$$S_n = \{z: z = \phi[(k - \tfrac{1}{2})/Nn + \tau], k = 1, 2,..., Nn\}, \qquad n = 1, 2,... .$$

We summarize our results in

Theorem 2. *Let Γ be an admissible Jordan curve. Then there is a set $L_n[\cdot; z]$ of Lagrange interpolation operators for which* (1) *holds for all $f \in \mathscr{C}(\Gamma)$, $z \in$* Int Γ, *uniformly on closed subsets of* Int Γ.

Moreover, if the set of corners $\{p_j\}$ is contained in

$$\{z: z = \phi[(k/N) + \tau], k = 1, 2,..., N\},$$

then one can take $L_n[f; \cdot]$ to be the polynomial of degree at most $Nn - 1$ which interpolates to f on the set $\{z: z = \phi[(k - \frac{1}{2})/nN + \tau], k = 1, 2,..., nN\}$.

It is clear that a certain flexibility exists in the selection of the sets S_n. That is, if S_n generates polynomials which satisfy (1) for all $f \in \mathscr{C}(\Gamma)$, and $\tilde{S}_n$ consists of a set $\{\tilde{z}_{nk}\}$ with $\tilde{z}_{nk}$ "close enough" to z_{nk}, then $\tilde{S}_n$ must also generate convergent polynomials. However, Lemma 4 is quite delicate and, in this treatment, essential to our conclusions. It is not clear that one can formulate a precise yet simple criterion for "close enough" in this context as one can for the analogous question for smooth curves [6].

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